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Canonical quantization of theories containing fractional powers of the d'Alembertian operator*

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Abstract. We present a canonical formulation for theories whose actions contain non-integer powers of the d'Alembertian operator and which were recently shown to play a central role in $(2+1)$-dimensional bosonization. We show that these theories possess an infinite number of constraints and use the Dirac method in order to obtain the classical brackets. The causal and classical Green functions are obtained and their meaning in terms of field expectation values is discussed. The Wightman functions are introduced and shown to lead to the microcausality principle. A mode expansion for the field is obtained. This permits the reobtention of the Wightman functions as vacuum expectation values of products of the basic fields. Creation and annihilation operators are naturally introduced but, as shown, they are not related to definite mass particle states. This is also confirmed by the spectral decomposition of the Wightman functions.

1. Introduction

It has been recently shown that the massless Dirac fermion field can be mapped to a vector (gauge) field which has the interesting property of presenting a square root of the d'Alembertian in its action [1, section 6]. That work motivates the present study of this kind of actions and in particular its canonical quantization.

There has long been an interest in non-local theories, as can be seen from the studies of Fokker and Feynman and Wheeler [2] on non-instantaneous interaction at a distance between particles in this first half of the century. In the 1950s there was a great deal of interest in non-local actions, as they were expected to be solutions to the infinities of field theory. Along this line of study we mention the Kristensen–Moller model [3] and the work by Pais and Uhlembeck [4] and others [5]. Further, there has been a great deal of work done by Efimov and coworkers up to recent times [6], addressing themes like strong interactions, anomalies and the avoidance of monopoles in GUT models. Recently, there has been a revival of interest in non-local actions, partly due to the appearance of non-local vertices in string field theory [7]. The Green functions of arbitrary powers of the d'Alembertian operator in an arbitrary dimension are also being studied by Giambiagi and Bollini [17], who apply a method due to Riesz [12] for the obtention of the Green functions and discuss the Huygens principle in arbitrary dimensions.

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Our work departs from most of the previous literature in that the non-locality is present in the kinetic term of the action rather than being due to the interaction. One of the first problems that soon afflicted the Hamiltonian treatment of non-local theories was the observation that the Poisson brackets between the field and all its derivatives are zero. Here we show that properly considering the momenta as constraints leads to non-zero Dirac brackets. Our treatment is also different from the perturbative treatment of non-local theories presented in [8] in that we do not have an expansion parameter and no reduction of order 2 occurs. Instead, we will consider the whole series expansion and sum it. Further, our work differs from [17] as our aim is to verify the applicability of canonical methods when discussing a definite class of non-local theories.

The organization of the article is as follows. In section 2, the classical canonical formulation is applied and the Dirac brackets are obtained. The Hamiltonian is derived and shown to lead to the Euler–Lagrange equation as a canonical equation. The third section introduces the quantization of the theory with the study of the causal Green functions. We argue that the Feynman prescription is associated with the vacuum expectation value of the time ordered product of fields, as in local cases. The classical retarded and advanced Green functions are also evaluated. For $\alpha = \frac{1}{2}$ these are shown to satisfy the Huygens principle. In the fourth section we obtain the Pauli–Jordan and Wightman functions and show that the former satisfy the microcausality condition, in spite of the fact that the Lagrangian is non-local. We also derive the spectral representation for the Wightman function. Section 5 is devoted to the obtention of a mode expansion for the basic field in terms of creation and annihilation operators. These, however are not related to particle states with a definite mass, as is also indicated by the spectral decomposition. Finally, in section 6 we summarize our conclusions and comment on the applications to bosonization in $2+1$ dimensions.

2. Canonical formalism for theories with pseudodifferential operators

In this section we present the canonical formalism suited to theories envolving non-localities associated with pseudodifferential operators of the type $\Box^{1-\alpha}$. If $1-\alpha$ is not a positive integer we are obviously dealing with an equation of motion that is not differential, but will involve a generalization of differential operators. In this work, we will consider the theory of a single scalar:

$$S = \frac{1}{2}\int \partial^{\mu}\phi(-\Box)^{-\alpha}\partial_{\mu}\phi \, \mathrm{d}^3x \tag{2.1}$$

where $(-\Box)^{-\alpha} = \int[\mathrm{d}^3k/(2\pi)^3](k^2)^{-\alpha}\mathrm{e}^{\mathrm{i}kx}$, and we are considering a $(2+1)$-dimensional space, keeping in mind the application to bosonization.

We will consider the more general case in which $(-\Box)^{-\alpha}$ is exchanged by $(-\Box+\chi)^{-\alpha}$, where χ will eventually be sent to zero at the end. With this we can take the formal series expansion in powers of the d'Alembertian resulting in an action with all derivatives present

$$S = \frac{1}{2}\int \sum_{n=0}^{\infty} a_n \, \partial^{\nu}\phi\Box^n\partial_{\nu}\phi \, \mathrm{d}^3x. \tag{2.2}$$

In this expression $a_n \propto \chi^{(-\alpha+n)}$.

There is a well-known method for the canonical treatment of higher derivative theories [9, 10]. Having realized that our theory can be considered as such we can apply the results of that treatment to our case. We will not present a full review of that treatment here, but stress its most important features. Assume the Lagrangian contains derivatives up to the order N. In this method the field ϕ and its time derivatives, $\dot{\phi}^{(n)}$, up to the order $N-1$ are taken as independent variables defining the coordinate space ($\dot{\phi}^{(n)}$ represents the nth time derivative of ϕ). The momenta are defined in such a way that the variation of the action, under the assumption of Lagrangian equations of motion, is equal to the ordinary expression [10]

$$\Delta S = \sum_{n=0}^{N-1} \int \frac{\mathrm{d}}{\mathrm{d}t}(\Pi^{(n)} \delta \dot{\phi}^{(n)})\, \mathrm{d}^3 x. \tag{2.3}$$

The $\dot{\phi}^{(n)}$ together with the associated momenta $\Pi^{(n)}$ define the phase space .

The Lagrangian equations are obtained with fixed variations of the fields and derivatives on the boundaries of the integration region in the action. For simplicity these can be taken as the infinite space in the extremal times. Poisson brackets are defined in such a way that each variable of the coordinate space obeys a canonical relation with its respective momenta all other brackets being zero. The Hamiltonian is defined by the Legendre transform of the Lagrangean, taking into account all derivatives of the fields composing the coordinate space. The canonical equations of motion then result from the Poisson bracket of each variable and the Hamiltonian, just as in the first order case.

This scheme has been applied to a wide range of higher-derivative theories [10], having been used together with the Dirac method whenever constraints are present. We are going to apply this method to the infinite derivative case and study its consequences, verifying that it really gives sound results.

One has two strategies to follow according to whether one truncates, or not, the alluded series expansion. Our strategy in this work is to take the infinite case and deal non-pertubatively with the whole series and try to obtain results under the assumption of its summation. In this approach the fact that each term of the series diverges when the auxiliary parameter χ is finally set to zero is by no means a catastrophe as long as the sum is well behaved in this limit. This is analogous to what happens in the computation of the effective potential when one expands around a massless theory and the whole sum is finite, in spite of the fact that each term is infrared divergent [11].

We obtain first the Euler–Lagrange equation. Varying equation (2.2) with respect to the field and its derivatives we get

$$\sum_{n=0}^{\infty} a_n \Box^{n+1} \phi = 0. \tag{2.4}$$

Summing the series we have the non-local equation of motion

$$\Box(-\Box + \chi)^{-\alpha} \phi = 0. \tag{2.5}$$

Let us perform now a canonical transformation which is legitimate for finite N, and we assume also holds for the infinite case. As we will see this will result in an enormous simplification. Let us take as independent variables the field ϕ together with all of its derivatives, which, for later convenience, we rearrange so that the new independent variables are $\phi_n = \Box^n \phi$ and $\dot{\phi}_n = \Box^n \dot{\phi}$. The momenta associated with these new variables are given by the variation of the action around a solution of the equation of motion

$$\Delta S = \int d^3x \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}(\Pi_n \delta\phi_n + P_n \delta\dot{\phi}_n). \tag{2.6}$$

Taking equation (2.6), we obtain

$$\Pi_n = \tfrac{1}{2} \sum_{j=n}^{\infty} (\dot{\phi}_{j-n} a_j + \delta_{n,0} \dot{\phi}_j a_j) \tag{2.7}$$

$$P_n = -\tfrac{1}{2} \sum_{j=n+1}^{\infty} \phi_{j-n} a_j. \tag{2.8}$$

In the truncated formalism one can see that some of the momenta are not independent variables as they can be expressed as linear combinations of variables in the coordinate space. In our case all of them are of this sort, as the coordinate space encompasses all the derivatives of the field. The momenta definitions have to be seen as constraints. We have thus found the primary constraints of our system. In order to see if they are first or second class we calculate the Dirac matrix between the constraints, which are

$$\begin{aligned} \chi_n &= \Pi_n - `\Pi_n\text{'} \\ \lambda_n &= P_n - `P_n\text{'}. \end{aligned} \tag{2.9}$$

Here 'Π_n' ('P_n') are the explicit expressions for the momenta in terms of coordinates (equations (2.7) and (2.8)).

The computation of the Dirac matrix is straightforward as long as one deals carefully with the infinite summations present. The result has the structure

$$D = \begin{bmatrix} \{\chi_n(x), \chi_m(y)\} & \{\chi_n(x), \lambda_m(y)\} \\ \{\lambda_n(x), \chi_m(y)\} & \{\lambda_n(x), \lambda_m(y)\} \end{bmatrix} = \begin{bmatrix} 0 & B_{n,m}(x,y) \\ -B_{n,m}(x,y) & 0 \end{bmatrix} \tag{2.10}$$

with

$$B_{n,m}(x,y) = \{\chi_n(x), \lambda_m(y)\} = -a_{n+m}\delta^2(x-y). \tag{2.11}$$

As a working hypothesis we assume that this matrix is not singular. It follows that there will be no secondary constraints. The evolution of primary constraints by means of the total Hamiltonian will only fix the Lagrange multipliers and not imply new constraints. In the finite case, i.e. truncating the series, one would also obtain some momenta expressed as combinations of variables of the coordinate space. In that case the finite number of independent variables allows one to infer the non-singular character of the Dirac matrix. The existence of the inverse of the Dirac matrix between the primary constraints is thus expected from the experience with the finite case.

The obtention of the explicit form of D^{-1} is certainly not an easy task. We are nevertheless going to see that in spite of this we can characterize the structure of the model without its explicit knowledge.

We introduce then the formal inverse of the Dirac matrix,

$$D^{-1}(x,y) = \begin{bmatrix} 0 & -B^{-1}_{n,m}(x,y) \\ B^{-1}_{n,m}(x,y) & 0 \end{bmatrix} \tag{2.12}$$

with

$$\int \mathrm{d}^2 y \sum_{j=0}^{\infty} B^{-1}_{m,j}(x,y)\; B_{j,n}(y,z) \equiv \delta_{n,m}\delta^2(x-z). \tag{2.13}$$

The calculation of the Dirac brackets is now straightforward, yielding the result

$$\begin{aligned} \{\phi_n(x), \phi_m(y)\}^* &= \{\dot\phi_n(x), \dot\phi_m(y)\}^* = 0 \\ \{\phi_n(x), \dot\phi_m(y)\}^* &= -B^{-1}_{n,m}(x,y). \end{aligned} \tag{2.14}$$

It is here that the sensible choice of independent variables comes into play. If the Dirac matrix did not have the simple form we would be unable able to guess its inverse form and what follows would be much more difficult.

We introduce now one more variable, $\rho = (-\Box + \chi)^{-\alpha}\phi$. Let us compute the Dirac bracket between ϕ and $\dot\rho$. Expanding the operator $(-\Box + \chi)^{-\alpha}$, in accord with our formalism we obtain

$$\begin{aligned} \{\phi(x), \dot\rho(y)\}^* &= \sum_{n=0}^{\infty} a_n \{\phi(x), \dot\phi_n(y)\}^* = -\sum_{n=0}^{\infty} a_n B^{-1}_{0,n}(x,y) \\ &= \int \mathrm{d}^2 z \sum_{n=0}^{\infty} B_{n,0}(z,y)\; B^{-1}_{0,n}(x,z) \\ &= \delta^2(x-y). \end{aligned} \tag{2.15}$$

In the same way we can obtain the whole structure

$$\begin{aligned} \{\phi(x), \dot\rho(y)\}^* &= -\{\dot\phi(x), \rho(y)\}^* = \delta^2(x-y) \\ \{\phi(x), \rho(y)\}^* &= 0. \end{aligned} \tag{2.16}$$

We stress that the above brackets are the result of infinite summations of the series. So we understand that they have a more fundamental meaning than relations (2.14). For instance, when finally $\chi \to 0$ the above result is unchanged while in relations (2.14) the Dirac matrix will diverge. We will reobtain relations (2.16) also in the quantum treatment; then they will be the result of the computation of the commutation relations by means of the explicit knowledge of the expectation value of the product of two fields. We are going to see also that this result has important consequences when applied to bosonization.

We can now consider dynamics. The Hamiltonian is defined just by the Legendre transform of the Lagrangian:

$$H = \int \mathrm{d}^2x \sum_{n=0}^{\infty} (\Pi_n \dot{\phi}_n + P_n \ddot{\phi}_n)\mathrm{d}^2x - L . \tag{2.17}$$

This can be worked out to the form

$$H = \int d^2x \sum_{n=0}^{\infty} \tfrac{1}{2}\Big(\partial_\mu \phi_n(x) \sum_{j=0}^{\infty} \partial_\mu \phi_j(x) a_{n+j} - \phi_{n+1}(x) \sum_{j=1}^{\infty} \phi_j(x) a_{n+j} \Big) . \tag{2.18}$$

We can now calculate the canonical equations of motion $\dot{A} = \{A, H\}$. When applied to ϕ_n the result is an identity: $\dot{\phi}_n = \dot{\phi}_n$. Observe that this is just what occurs in the local case to the evolution of ϕ. For the evolution of $\dot{\rho}(= \sum a_n \dot{\phi}_n)$ one obtains

$$\begin{aligned}
\ddot{\rho}(x) &= \sum_{n=0}^{\infty} a_n \{\dot{\phi}_n(x), H\}^* \\
&= \sum_{n=0}^{\infty} a_n \int \mathrm{d}^2y\, \mathrm{d}^2z \Big(\sum_{k,j=0}^{\infty} \nabla^2 \phi_k(y) B_{k,j}(y,z)\ B^{-1}_{j,n}(z,x) \\
&\qquad - \sum_{k,j=1}^{\infty} a_{k+j-1} \phi_k(y) B^{-1}_{j,n}(z,x) \delta(z-y) \Big) \\
&= \sum_{n=0}^{\infty} a_n \nabla^2 \phi_n(x) - \int \mathrm{d}^2y \sum_{k,j=1}^{\infty} a_{j+k-1} \phi_{k+1}(y) \sum_{n=0}^{\infty} B^{-1}_{j,n}(y,x) a_n \\
&= \nabla^2 \rho(x) + \sum_{k,j=1}^{\infty} a_{j+k-1} \phi_k(x)\ \delta_{j,0} = \nabla^2 \rho(x).
\end{aligned} \tag{2.19}$$

The last equation is just the Euler–Lagrange equation reobtained by Hamiltonian methods. Note that it can be rewritten as $(-\Box)^{1-\alpha}\phi = 0$, when χ is set to zero. Thus, we reach the conclusion that the formalism proposed has been able to perform its task, giving not only a sound canonical structure but also the correct equation of motion. One might argue that this formalism is somewhat unuseful, as it requires knowledge of all time derivatives of the field in order to have information on the evolution of the field. It says nevertheless that not all derivatives are finally independent but are related by dynamics. The fact that the evolution equation requires knowledge of all field derivatives is a manifestation of its non-local character as long as the behaviour of the field on a spacetime point depends on its values in an extended region of spacetime, and not just in the vicinity.

Let us now once more consider the equation of motion in its series expansion form, (equation (2.4)). We point out that the equation of motion has a different character from the usual cases. Namely, we observe that, as all derivatives belong to coordinate space, the equation of motion is indeed a constraint. We emphasize that this constraint does not appear as a secondary constraint from the Dirac algorithim in the usual sense. We have already seen that the evolution of the primary constraints only fixes the Lagrange multipliers. We have then to consider the following question: are these constraints respected by the Dirac brackets? One becomes easily convinced of the negative answer. This is expected as we have not considered them when establishing the Dirac brackets. One can envisage the possibility of pursuing further the Dirac method and adding the new constraints to the Dirac algorithim.

The new constraints should be taken as an infinite number of equations. Besides $\xi = \sum_{n=0}^{\infty} a_n \Box^{n+1}\phi$ we also have $\xi_n = \Box^n \xi$ and $\dot{\xi}_n = \Box^n \dot{\xi}$. It is clear what we should do now: calculate the Dirac matrix between the new constraints, using our hitherto obtained Dirac brackets, invert it, and redefine the Dirac brackets. One will find in this procedure the same kind of difficulties occurring previously. The new Dirac matrix is

$$D'(x,y) = \begin{bmatrix} \{\xi_i(x), \xi_j(y)\}^* & \{\xi_i(x), \dot{\xi}_j(y)\}^* \\ \{\dot{\xi}_i(x), \xi_j(y)\}^* & \{\dot{\xi}_i(x), \dot{\xi}_j(y)\}^* \end{bmatrix} = \begin{bmatrix} 0 & C_{i,j}(x,y) \\ -C_{i,j}(x,y) & 0 \end{bmatrix}$$

with

$$C_{i,j}(x,y) = -\sum_{k,p=0}^{\infty} a_k a_p B^{-1}_{i+k+1,j+p+1}(x,y). \tag{2.20}$$

Once again its non- singular character must be assumed. The inverse of D' must be defined by the formal properties

$$D'^{-1}(x,y) = \begin{bmatrix} 0 & -C^{-1}_{i,j}(x,y) \\ C^{-1}_{i,j}(x,y) & 0 \end{bmatrix}$$

with

$$-\int \mathrm{d}^2 z \sum_{i,k,p=0}^{\infty} a_k a_p C^{-1}_{n,i}(x,z) B^{-1}_{i+k+1,j+p+1}(z,y) \equiv \delta_{n,j}\delta^2(x-y)$$

and

$$-\int \mathrm{d}^2 z \sum_{j,k,p=0}^{\infty} a_k a_p B^{-1}_{i+k+1,j+p+1}(x,z) C^{-1}_{j,n}(z,y) \equiv \delta_{i,n}\delta^2(x-y). \tag{2.21}$$

The new Dirac brackets will require a redefinition of the unknown constants in relations (2.14),

$$\{\dot{\phi}_n(x), \phi_m(y)\}^{**} = B^{-1}_{n,m}(x,y) + \int \mathrm{d}^2 z\, \mathrm{d}^2 z' \sum_{k,\ell,i,j,=0}^{\infty} a_i a_j \times B^{-1}_{n,k+i+1}(x,z) C^{-1}_{k,\ell}(z,z')\ B^{-1}_{j+\ell+1,m}(z',y)$$

still with

$$\{\phi_n(x), \phi_m(y)\}^{**} = \{\dot{\phi}_n(x), \dot{\phi}_m(y)\}^{**} = 0. \tag{2.22}$$

With these new Dirac brackets we can once again compute the Dirac brackets between the fields ϕ and ρ. Using equations (2.21) and (2.22) we obtain that equations (2.16) remain unchanged.

Furthermore, in the same way we performed the calculation of equations (2.16), we can now obtain the Dirac brackets between the new variables $(-\Box+\chi)\ \phi$ and $(-\Box+\chi)^{-1-\alpha}\dot{\phi}$. We obtain the result

$$\{(-\Box+\chi)\ \phi(x),\ (-\Box+\chi)^{-1-\alpha}\dot{\phi}(y)\}^{**} = \delta^2(x-y). \tag{2.23}$$

In the case where we have an interaction term like $V(\phi)$ the above treatment will imply the equation of motion: $-\Box(-\Box+\chi)^{-\alpha}\phi = V'(\phi)$. This is just the expected Euler–Lagrange equation.

We should also comment on what happens when $-\alpha$ is an integer or zero. In this case we will not be dealing with non-local theories but with a local one. For $\alpha = 0$, for instance, the action still has the general form of equation (2.2) but with $a_n = 0$ for $n > 0$. We can see, nevertheless, that the Dirac brackets which result from the infinite summation of series such as $\{\phi, \dot{\rho}\}^*$ go smoothly to the ones obtained from the treatment of the resulting model. In the case $\alpha = 0$, ρ is identified with ϕ itself and relations (2.16) are just the usual ones.

3. Quantization: Green functions

3.1. The causal Green functions

Let us now turn our attention to the quantum case. We have an equation of motion, a quadratic Lagrangian (2.1), and a classical canonical structure to guide us. We will start by computing the relevant Green functions and analysing their physical meaning in terms of expectation values of products of fields. We are simply going to illustrate that the quantum treatment of these non-local free theories will still present interesting features. In some sense these theories behave more like interacting rather than free theories.

First of all, we calculate the expectation values of the time ordered product of two fields. As the Lagrangian is quadratic, the causal Green function is just $(-\Box)^{-1+\alpha}$. We have to ascribe a prescription to the contour not of a pole but of the branch point and the cut associated with the root. Adopting the Feynman prescription, $k^2 \to k^2 + \mathrm{i}\varepsilon$, we get the result (appendix 1)

$$D_c^\alpha(x) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k \frac{\mathrm{e}^{-\mathrm{i}kx}}{(k^2+\mathrm{i}\varepsilon)^{1-\alpha}} = \beta_\alpha(t^2 - r^2 - \mathrm{i}\varepsilon)^{-(\alpha+1/2)} \tag{3.1}$$

with $t = x^0$, $r = |\boldsymbol{x}|$, and

$$\beta_\alpha = -(2\pi)^{-3/2}2^{2\alpha-1/2}\Gamma(\alpha+\tfrac{1}{2})/\Gamma(1-\alpha).$$

The cut has been chosen in the positive k^2 (or x^2) real axis (figure 1).

We are now in position to ask the following question: is a field with commutation rules taken from the above Dirac brackets a quantum solution to the non-local equation of motion derived from the action? A way to answer this question is to find if the expectation value of the time ordered product of two fields is just the causal function evaluated above. To find this out we apply the operator $(-\Box+\chi)^{1-\alpha}$ to such an expectation value in order to see if it results in a tridimensional delta function. In this computation we are going to use the equal time commutation rules taken from the

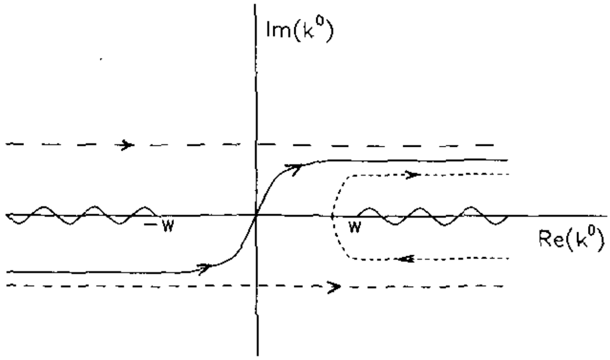

Figure 1. Integration paths for the computation of the Green and Wightman functions: — — — $D_{\rm adv}$; —— D_c; - - - - - D_+; - - - $D_{\rm ret}$.

(first) Dirac brackets of the previous section by use of the correspondence principle. First note that, using equations (2.14),

$$\begin{aligned}\Box T\phi(x)\ \phi(y) &= \Box(\theta(x^0-y^0)\phi(x)\phi(y)+\theta(y^0-x^0)\phi(y)\phi(x))\\ &= \partial_\nu(T\partial^\nu\phi(x)\ \phi(y)) \\ &= T\Box\phi(x)\phi(y)+\mathrm{i}\delta(x^0-y^0)B^{-1}_{0,0}(x,y)\end{aligned}\tag{3.2}$$

By iterative application of the d'Alambertian and use of equations (2.14), we obtain

$$\Box^n T\phi(x)\phi(y) = T\Box^n\phi(x)\phi(y)+\mathrm{i}\sum_{j=0}^{n-1}\Box^j(\delta(x^0-y^0)B^{-1}_{n-j-1,0}(x,y))\tag{3.3}$$

Following our previous steps, we expand the operator $\Box/(-\Box+\chi)^\alpha$ in powers of $\Box$ and apply it to the time product:

$$\begin{aligned}\Box(-\Box+\chi)^{-\alpha}T\phi(x)\phi(y) &= \sum_{n=0}^{\infty}a_n\Box^{n+1}T\phi(x)\phi(y)\\ &= \sum_{n=0}^{\infty}a_n\ T\Box^{n+1}\phi(x)\ \phi(y)+\mathrm{i}\sum_{n=0}^{\infty}\sum_{m=0}^{n}a_n\Box^m\delta(x^0-y^0)B^{-1}_{n-m,0}(x,y).\end{aligned}\tag{3.4}$$

By redefining the summation indices we obtain the following expression for the last term in the previous formula:

$$\begin{aligned}&\mathrm{i}\sum_{m=0}^{\infty}\Box^m\left(\delta(x^0-y^0)\sum_{n=0}^{\infty}B^{-1}_{n,0}(x,y)a_{n+m}\right)\\ &\quad= -\mathrm{i}\sum_{m=0}^{\infty}\Box^m\left(\delta(x^0-y^0)\int \mathrm{d}^2z\sum_{n=0}^{\infty}B^{-1}_{n,0}(z,y)B_{m,n}(x,z)\right)\\ &\quad= -\mathrm{i}\sum_{m=0}^{\infty}\Box^m\left(\delta(x^0-y^0)\delta_{m,0}\delta(\boldsymbol{x}-\boldsymbol{y})\right)\\ &\quad= -\mathrm{i}\ \delta^3(x-y).\end{aligned}\tag{3.5}$$

Due to the field equation $\Box(-\Box+\chi)^{-\alpha}\phi(x)=0$, the first term of the same expression is zero:

$$\sum_{n=0}^{\infty} a_n T\Box^{n+1}\phi(x)\ \phi(y)=T\Box\rho(x)\phi(y)=0. \tag{3.6}$$

Summing up, we have shown that

$$\Box(-\Box+\chi)^{-\alpha}\langle T\phi(x)\phi(y)\rangle=-\mathrm{i}\delta^3(x-y). \tag{3.7}$$

This shows that the first Dirac brackets obtained in the previous section are enough to attribute to the function $(-\mathrm{i})\langle T\phi\phi\rangle$ the character of the inverse of the operator appearing in the action. It is interesting to note that a prescription is already needed in the operator definition and not only in the Green function.

Does the above reasoning also hold if one uses the double-star Dirac brackets? Indeed, if one performs the calculation with the new Dirac brackets one will see that instead of $\langle T\phi\phi\rangle$ the Dirac delta function will result for the non-local operator applied to $(-\Box+\chi)^{\alpha}\langle T\rho\phi\rangle=(-\Box+\chi)^{\alpha}\langle T(-\Box+\chi)^{-\alpha}\phi\phi\rangle$. Using the single star brackets both functions will have this property. Indeed, the above calculation actually means that $(-\Box+\chi)^{-\alpha}T\phi\phi=T(-\Box+\chi)^{-\alpha}\phi\phi$ for one-star Dirac brackets (this property is easily verifyied when $-\alpha$ is a positive integer, i.e. for local cases). So we see that there is an inconsistency between the implementation of the equations of motion as constraints and the Green function character of the expectation value of the time ordered products of the basic field ϕ. We will return to this question in the next section and keep interpreting $\langle T\phi\phi\rangle$ as the causal Green function.

We add one more comment in relation to this. We could also have started with the Lagrangian $\mathcal{L}=\phi(-\Box+\chi)^{1-\alpha}\phi$. The single-star Dirac brackets would be subtly changed ($B_{n,m}\to B'_{n,m}=-a_{n+m}+\chi a_{n+m+1}$) and the basic Dirac bracket $\{\phi,\dot{\rho}\}^*=\mathrm{i}\delta^2(x-y)$ would still follow, but only if $1-\alpha>0$, i.e. positive powers of the d'Alembertian. The equation defining the propagator, with single-star Dirac brackets, would be $(-\Box+\chi)^{1-\alpha}\langle T\phi(x)\phi(y)\rangle=-\mathrm{i}\delta^3(x-y)$. It is tempting to associate this with the fact that only for $1-\alpha>0$ does the operator $(-\Box+\chi)^{1-\alpha}$ mean generalization of the differential operators $\Box^n$ [12].

3.2. The Classical functions

In appendix 2 we calculate the classical Green functions (retarded and advanced)

$$\begin{aligned} D^{\alpha}_{\mathrm{ret}} &= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3k\mathrm{e}^{-\mathrm{i}kx}}{[(k^0+\mathrm{i}\varepsilon)^2-\boldsymbol{k}^2]^{1-\alpha}} \\ &= -\beta_\alpha\theta(x^0)[(t^2-r^2+\mathrm{i}\varepsilon)^{-(\alpha+1/2)}-(t^2-r^2-\mathrm{i}\varepsilon)^{-(\alpha+1/2)}] \end{aligned} \tag{3.8}$$

$$\begin{aligned} D^{\alpha}_{\mathrm{adv}} &= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3k\mathrm{e}^{-\mathrm{i}kx}}{[(k^0-\mathrm{i}\varepsilon)^2-\boldsymbol{k}^2]^{1-\alpha}} \\ &= -\beta_\alpha\theta(-x^0)[(t^2-r^2+\mathrm{i}\varepsilon)^{-(\alpha+1/2)}-(t^2-r^2-\mathrm{i}\varepsilon)^{-(\alpha+1/2)}]. \end{aligned} \tag{3.9}$$

These distributions have been studied by Riesz [12] who showed that they are indeed inverses of the powers of the d'Alembertian when $1-\alpha$ is a positive intenger.

In other cases they are identified with the $\Box^{1-\alpha}$ operator or its inverse, according to whether $\alpha - 1$ is greater or smaller then zero [3]. Note that in the $\varepsilon \to 0$ limit only the discontinuity along the cuts will contribute. It follows, noting that the cut lies on the positive x^2 axis, that the classical functions have the interesting property of respecting causality although the Lagrangian itself does not suggest it! Note further that for $\alpha = \frac{1}{2}$, in particular, the expression within brackets in equations (3.8) and (3.9) becomes a delta function and we see that the classical functions have support on the light-cone surface, implying that not only causality but also the Huygens principle would be respected. This result has also been obtained in [17] by use of the Riesz method. It is interesting to remark that the $\alpha = \frac{1}{2}$ case is precisely the one relevant for the bosonization [1].

4. Quantization: the Pauli–Jordan and Wightman functions

In order to have a complete description of the theory we have to define the Pauli–Jordan function and its negative- and positive-frequency parts. We are going to ascribe a definite meaning to these functions that furnish the expectation value of the ordinary product of two fields. Notice that we need a function that can be split into positive and negative frequencies. Further, it should be a solution of the equation of motion and coincide with the local function for $\alpha = 0$. We define

$$D^\alpha_{\mathrm{PJ}} = D^\alpha_+ - D^\alpha_- \qquad \text{and} \qquad D^\alpha_-(x) = D^\alpha_+(-x). \tag{4.1}$$

We take as the Wightman functions $D^\alpha_+(x)$ the following:

$$D^\alpha_+(x) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\,\theta(k^0)[1/(k^2+\mathrm{i}\varepsilon)^{1-\alpha} - 1/(k^2-\mathrm{i}\varepsilon)^{1-\alpha}]\mathrm{e}^{-\mathrm{i}kx} \tag{4.2}$$

where the cut is in the positive real k^2 axis (figure 1).

Note that when $\alpha = 0$ the last expression takes the usual form. In this case the difference in the integrand is just a representation of the delta function. Note also that due to the cut on the positive k^0 axis the function comprises only positive frequencies. This last point deserves to be stressed: these functions present a decomposition into positive and negative frequencies that are respected by Lorentz transformations.

With the above prescription we find

$$D^\alpha_+ = \beta_\alpha[(t-\mathrm{i}\varepsilon)^2 - r^2]^{-(\alpha+1/2)}. \tag{4.3}$$

This strongly suggests a generalization of the ordinary case. Indeed, it can be shown that when $\alpha = -n$ $(n = 1, 2, \ldots)$ the above function may be obtained from the treatment of the resulting higher (finite)-order theories. These functions represent an analytic interpolation between higher-order theories.

Can these functions really be associated with the expectation value of the simple product of two fields? First we note that with the D_+ we obtain the equal time commutation relation read from the Dirac brackets by operator product expansion. One can easily see that the commutators $[\Box^n\phi, \Box^n\phi]$ and $[\Box^n\dot{\phi}, \Box^m\dot{\phi}]$ vanish, while $[\Box^m\phi, \Box^m\dot{\phi}]$ may be non- zero. This is obtained by taking the difference between $D^+_\alpha(x-y)$ and $D^+_\alpha(y-x)$, associated respectively with $\langle\phi(x)\phi(y)\rangle$ and $\langle\phi(y)\phi(x)\rangle$, and applying the appropriate powers of $\Box$. As we deal with equal time commutators we have to make $x^0 - y^0 = 0$ in the final expressions. The commutator of ϕ and $\dot{\phi}$, for instance, is seen to be

$$[\phi(x), \dot{\phi}(y)]_{x^0=y^0} = 4(\alpha + \tfrac{1}{2})\mathrm{e}^{-\mathrm{i}\pi(\alpha+1/2)} \frac{(-\mathrm{i}\varepsilon)}{[(\boldsymbol{x}-\boldsymbol{y})^2 + \varepsilon^2]^{-\alpha+3/2}}. \tag{4.4}$$

The above expression can be seen as a sequence of functions associated with a distribution. It is clear that it is different from zero only for $\boldsymbol{x} = \boldsymbol{y}$. It is easy to show, by integrating in space, that they are more singular around $\boldsymbol{x} = \boldsymbol{y}$ than a delta function for $\alpha > 0$. Renormalizing the product by multiplication of the fields by ε^α one obtains a bidimensional delta function sequence. In this way we obtain all the zero Dirac brackets as commutation rules. The non-zero ones have the forementioned singular character.

Moreover, if we identify $\langle\phi(x)\rho(y)\rangle$ with $D^0_+(x-y)$, as we will soon argue (see equation (4.12)) it will follow that the commutator of ϕ and $\dot{\rho}$ is the delta function, without resort to any infinite renormalization in the point-splitting calculation.

Further, if the D_+ functions are really the above expectation values, then they *might* themselves satisfy the equations of motion. One possible way to verify this would be simply to apply the operator $(-\Box)^{1-\alpha}$ inside the Fourier transformation integral. Then one would deal with products of the type $k^{2\beta}(1/(k^2+\mathrm{i}\varepsilon)^{1-\alpha} - 1/(k^2 - \mathrm{i}\varepsilon)^{1-\alpha})$. If these products are redefined as $(1/(k^2+\mathrm{i}\varepsilon)^{1-\alpha-\beta} - 1/(k^2-\mathrm{i}\varepsilon)^{1-\alpha-\beta})$ one would obtain the expected equation of motion for an appropriate value of β, namely $\beta = 1-\alpha$.

An alternative and more instructive way of obtaining the equations of motion is through the use of the spectral decomposition, as we will show. Note that for non-integer α the D^α_+ functions can be reexpressed as

$$D^\alpha_+(x) = b_\alpha \int \frac{\mathrm{d}^3k}{(2\pi)^3} \mathrm{e}^{-\mathrm{i}kx}\theta(k^0)\ (k^2_+)^{\alpha-1} \tag{4.5}$$

where $(k^2_+)^{(\alpha-1)} = \theta(k^2)(k^2)^{(\alpha-1)}$ and $b_\alpha = 2\exp[\mathrm{i}\pi(\alpha - \frac{1}{2})]\sin[\pi(1-\alpha)]$.

We can introduce into the integrand the resolution of identity $\int_0^\infty \delta(k^2 - m^2)\mathrm{d}m^2 = 1$. Changing the order of integration to

$$\begin{aligned} D^\alpha_+(x) &= b_\alpha \int_0^\infty \mathrm{d}m^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{-\mathrm{i}kx}\ (m^2)^{\alpha-1}\theta(k^0)\delta(k^2-m^2) \\ &= b_\alpha \int_0^\infty \mathrm{d}m^2\ (m^2)^{\alpha-1} D^0_+(x,m) \end{aligned} \tag{4.6}$$

where we have introduced the massive local function

$$D^0_+(x,m) = \frac{1}{(2\pi)^3)} \int \mathrm{d}^3k \mathrm{e}^{-\mathrm{i}kx} \left(\frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon} - \frac{1}{k^2 - m^2 - \mathrm{i}\varepsilon}\right)\theta(k^0) \tag{4.7}$$

observe that it satisfies the eigenvalue equation

$$-\Box D^0_+(x,m) = m^2 D^0_+(x,m). \tag{4.8}$$

Equation (4.6) shows spectral decomposition of the D_+ functions in terms of local ones. It is seen that all the masses of the local functions contribute. We can now begin to understand the spectrum of our model. The important point to be stressed here is that the functions appearing in equation (4.6) are the local massive D_+ functions. In other words the expectation value of the product of two non-local fields can be expressed as the integral in the masses of the expectation value of the product of local massive fields.

We can now show how the equation of motion is satisfied by the $D_+^\alpha(x-y)$, i.e. by the expectation value of the simple product of two fields.

We have to apply $(-\Box)^{1-\alpha}$ to $D_+^\alpha(x)$. We first factorize the operator $(-\Box)^{1-\alpha} \to -\Box(-\Box)^{-\alpha+\delta/2}$, and apply the last factor first. Here δ is a regulator that will be set to zero afterwards. Using the spectral decomposition equation (4.6) and the eigenvalue equation (4.8), the application of the non-local operator is now transparent:

$$\begin{aligned}(-\Box)^{-\alpha+\delta/2}D_+^\alpha(x) &= b_\alpha \int_0^\infty \mathrm{d}m^2\,(m^2)^{\alpha-1}(-\Box)^{-\alpha+\delta/2}D_+^0(x,m)\\ &= b_\alpha \int_0^\infty \mathrm{d}m\, m^{\delta-1}\; D_+^0(x,m).\end{aligned} \tag{4.9}$$

Of course, an analytic continuation in the δ-plane is implied here. When the limit $\delta \to 0$ is taken, appears there a pole in δ:

$$\int_0^\infty \mathrm{d}m\, m^{\delta-1}D_+^0(x,m) = \int_0^1 \mathrm{d}m\, m^{\delta-1}D_+^0(x,m) + \int_1^\infty \mathrm{d}m\, m^{\delta-1}D_+^0(x,m). \tag{4.10}$$

The last factor is not singular so we leave it aside. The first one is singular in the limit $\delta \to 0$:

$$\begin{aligned}\lim_{\delta\to 0}\int_0^1 \mathrm{d}m\, m^{\delta-1}D_+^0(x,m) &= \lim_{\delta\to 0}\int_0^1 \mathrm{d}m\, m^{\delta-1}\;[D_+^0(x,0)+\mathcal{O}(m)]\\ &= \lim_{\delta\to 0}\left(\frac{1}{\delta}D_+^0(x,0) + \text{finite terms}\right)\end{aligned} \tag{4.11}$$

When δ goes to zero the singularity is removed by multiplication by δ, i.e. the expression is redefined by its residue. The result is the usual local function:

$$(-\Box)^{-\alpha}\; D_+^\alpha(x) \approx D_+^0(x). \tag{4.12}$$

The equation of motion then follows upon application of the remaining $(-\Box)$ factor. We stress here the privileged role played by the local zero-mass field in this calculation.

Furthermore, the above computation shows that $(-\Box)^\beta D_+^\alpha(x) = \text{constant} \cdot D_+^{\alpha+\beta}(x)$. When β is an integer the constant factor is 1. In this last case the same result can be obtained by direct differentiation.

We also point out that the following relations between the various functions still hold in the non-local case:

$$\begin{aligned}&D_{\mathrm{PJ}}^\alpha(x) = -D_{\mathrm{adv}}^\alpha(x) + D_{\mathrm{ret}}^\alpha(x)\\ &D_{\mathrm{ret}}^\alpha = \theta(x^0)D_{\mathrm{PJ}}^\alpha(x)\\ &D_{\mathrm{c}}^\alpha(x) = \theta(x^0)D_+^\alpha(x) + \theta(-x_0)D_-^\alpha(x)\\ &D_+^\alpha(x) = D_{\mathrm{c}}^\alpha(x) - D_{\mathrm{adv}}^\alpha(x).\end{aligned} \tag{4.13}$$

The above formulae show some similarities between local and non-local theories. In particular the causal functions can really be interpreted as expectation values of time ordered products of the fields. Also, the Pauli–Jordan functions, or field commutators, are expressible by use of the classical (retarded and advanced) functions. This implies that the fields, although satisfying a non-local evolution equation, still satisfy the microcausality principle. The main differences from the local theory are related to the spectrum as is indicated by the spectral representation (4.6). This will also be seen in the next section. We note also that, in the particular case $\alpha = \frac{1}{2}$, we obtain the quantum version of the Huygens principle: the fields commute out of the light-cone surface.

5. Mode expansion

In order to have a complete characterization of the quantum theory it is important to have a mode expansion which serves to expose the Fock space structure of the model. A natural procedure would be to search for a complete set of solutions of the equations of motion and associate with each of them a quantum excitation. In our case we do not possess such a set. It is not at all clear what are the complete solutions of a non-local equation such as $(-\Box)^{1-\alpha} f = 0$. We have seen, nevertheless, that the $D^{+}(x-y)$ functions do, in a definite way, satisfy these equations. We shall thus try to obtain the mode expansion based upon knowledge of this function.

In contrast to the local case the D_{+} function is not expressible as a bi-dimensional Fourier transform (see equation (4.2)). It does not contain a delta function. This suggests that we should try an expansion of the field with a tridimensional instead of a bidimensional integration. In addition, this we have already seen that the condition of microcausality holds in this case, assuring a unique decomposition between positive and negative frequencies. This suggests, then, the following expansion for $\phi(x)$

$$\phi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} (\mathrm{e}^{-\mathrm{i}kx} \phi(k) + \mathrm{e}^{+\mathrm{i}kx}\ \phi^{\dagger}(k)) \delta^{-\alpha}(k^2) \theta(k^0) \tag{5.1}$$

where we have defined $\delta^{\alpha}(k^2) = 1/(k^2 + \mathrm{i}\varepsilon)^{1-\alpha} - 1/(k^2 - \mathrm{i}\varepsilon)^{1-\alpha}$.

If the $\phi(k)(\phi^{\dagger}(k))$ are taken as annihilation (creation) operators of excitations with trimomentum k and ascribed the commutation rules $[\phi(k), \phi^{\dagger}(k')] = \delta(k - k')/\delta^{-\alpha}(k^2)$, we readily see that the operator has the expected correlation function, reproducing, up to a constant factor, the D_{+} function and consequently also the D_{PJ} and the D_{c} functions.

Let us remark on one point here. It is tempting to define $\phi(x) \equiv (-\Box)^{\alpha} \varphi(x)$, where φ is the usual local field, with its usual mode expansion. With this, the non-local equation of motion would follow from the local one: $(-\Box)^{1-\alpha}\phi(x) = -\Box\varphi(x) = 0$. The definition of $(-\Box)^{\alpha}\varphi(x)$ is nevertheless missing. In momentum space there appears $k^{2\alpha}\delta^{3}(k^2)$ and this does not have a definite meaning as a distribution.

In contrast to the local case the operator field not now does satisfy the equation of motion as a strong relation, but all expectation values of arbitrary local products (that do not involve non-local powers of the d'Alembertian among them) do satisfy it. The reason is simply that such expectation values involve the D_{+} functions which satisfy the proper equations of motion, in the above-explained sense.

Now what kind of excitations do $\phi(k)^{\dagger}$ create? First of all they do not represent different momenta of the same particle as they do not have a unique mass associated with them. We have already seen the appearance of such indeterminate mass when we performed the spectral decomposition of the functions D_+. Although the basic excitations are not particles, there is a non-local one which has a particle content. Indeed we have seen that in the calculation of the equation of motion the free massless scalar plays a privileged role. In other words, the expectation value of $\phi(-\Box)^{-\alpha}\phi$ is equal to the expectation values of two local fields. This suggests treating the local massless particle as a non-local excitation playing a privileged role in the Fock space.

Having this characterization of the Fock space we can now define normal ordering of any expression involving the fields in the usual way. This is crucial in applications to bosonization in three dimensions where we will have to deal with exponentials of the fields [1].

Let us point out what happens to the expansion in the local limit when $\alpha \to 0$. As already remarked, the $D_+^\alpha(x)$, which is obtainable with the help of expansion (5.1), goes to the local function by acquiring the factor $\delta(k^2)$ in momentum space. Indeed $\delta^n(k^2)$ is easily identified, up to constant factors, with derivatives of the delta function. The creation and annihilation operators, with k^2 different from zero do not contribute in this limit. This allows one to rewrite the expansion of the field in the usual way. In other words, the Fock space will be reduced, with the excitations with 'wrong' dispersion relations decoupling from the remaining physical sector.

6. Conclusion

In this work we considered the extension of the canonical treatment of systems with higher derivatives [1, 2] for the case where an infinite number of derivatives is present. We applied this method in the canonical formulation of non-local theories containing pseudodifferential operators in the kinetic term. The interest in such theories comes from the fact that they appear in the process of bosonization of the Dirac fermion field in $2 + 1$ dimensions [1].

Starting from the free Dirac Lagrangian, it is possible to show [1] that under a non-local transformation similar to that of Foldy and Wouthuysen one arrives at a Lagrangian involving two complex spin-zero fields whose actions are given by (2.1) (with $\alpha = \frac{1}{2}$). These spin zero fields, in their turn, may be expressed in terms of a vector field whose lagrangean again possesses the non-local operator $\Box^{1/2}$ [1]. These non-local theories were only treated within the functional integral formulation in [1], hence the interest in having an insight into their canonical quantization. This is what we pursued here in the case of a scalar field. In another work [15] we studied the case of vector field theories containing pseudodifferential operators. There we show that using the canonical quantization of the vector field and the representation of spin zero fields in terms of exponentials of it [1], the Wightman functions of the later can be obtained in terms of the Wightman functions of the former.

The canonical structure that we obtained in the present paper is characterized by the absence of independent momenta and by the fact that the equation of motion is ultimately a constraint between the variables in coordinate space. This fact is related to the lack of a proper initial-value (classical) problem since there is no finite number of derivatives defined on a spacelike surface determining the field values in all spacetime. Nevertheless, we have shown that a field quantization which paralells this classical structure is still possible. A quite remarkable fact is that the Pauli–Jordan commutation function respects the microcausality condition, in spite of the non-locality of the Lagrangian, and for the special case of $\alpha = \frac{1}{2}$ the classical Green functions obey the Huygens principle. An interesting feature is that the quantum field does not satisfy the equation of motion as a strong relation but as a weak one. We have also seen that a proper definition of vacuum expectation values of simple products of fields yields a mode expansion which leads to a natural definition of creation and annihilation operators and to the related concept of normal order. These creation and annihilation operators, however, are not related to definite mass states, as is also indicated by the spectral decomposition of the Wightman functions. In this sense, the non-local theory resembles an interacting one, in spite of being quadratic. An interesting fact is that the non-local field $\rho(x)$ creates states with a definite mass (equal to zero).

We finally comment that the lack of a particle content associated with the basic field is analogous to what happens in $(1+1)$-dimenisonal bosonization where the massless scalar field also does not have a well-defined particle content. The lack of a definite mass in this kind of non-local theory suggests that the basic fields should transform under a non-unitary representation of the Poincaré group. This deserves further investigation.

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Appendix 1

Let us compute here the Fourier transform in equation (3.1):

$$D_c^\alpha(x) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3k\,\mathrm{e}^{-\mathrm{i}kx}}{(k^2+\mathrm{i}\varepsilon)^{1-\alpha}}$$
$$= \frac{1}{(2\pi)^3}\int \mathrm{d}^2k\,\mathrm{e}^{+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0\,\mathrm{e}^{-\mathrm{i}k^0x^0}}{[\mathrm{i}k^0+\mathrm{i}w+\varepsilon]^{1-\alpha}[-\mathrm{i}k^0+\mathrm{i}w+\varepsilon]^{1-\alpha}}. \tag{A1.1}$$

Here and in the following, $w = |\boldsymbol{k}|$.

According to [13, p 321] this can be rewritten

$$D_c^\alpha(x) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2k\,\mathrm{e}^{+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}(2\varepsilon+2\mathrm{i}w)^{\alpha-1}\frac{|x^0|^{-\alpha}}{\Gamma(1-\alpha)}W_{0,\alpha-1/2}[2\mathrm{i}(w-\mathrm{i}\varepsilon)|x^0|]. \tag{A1.2}$$

The angular integration is easily performed [13, p 952], giving the result

$$D_{\mathrm{c}}^{\alpha}(x)=\frac{\mathrm{e}^{(\mathrm{i}\pi/2)(\alpha-1/2)}}{2\pi^{3/2}\Gamma(1-\alpha)}\left(\frac{(x^0)^2}{2}\right)^{1/2-\alpha}$$
$$\times\int_0^{\infty}J_0(w|\boldsymbol{x}|)\;K_{\alpha-1/2}[\mathrm{i}w(|x^0|-\mathrm{i}\varepsilon)]w^{\alpha+1/2}\mathrm{d}w \tag{A1.3}$$

where we have expressed $W_{0,\alpha-1/2}$ in terms of the Bessel function $K_{\alpha-1/2}$ [13, p 1062]. We also use the fact that $w>0$ to redefine ε. Using [14, p 365] the last integral is evaluated. Taking the limit $\varepsilon\to 0$ we obtain equation (3.1).

The same result has been obtained in [16, p 365]. Note, however, that our convention for the position of the cut differs from that in [16].

Appendix 2

Let us compute now the retarded function:

$$D_{\mathrm{ret}}^{\alpha}(x)=\frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3k\mathrm{e}^{-\mathrm{i}kx}}{[(k^0+\mathrm{i}\varepsilon)^2-k^2]^{1-\alpha}}$$
$$=\frac{1}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\pi(\alpha-1)}\int\mathrm{d}^2k\mathrm{e}^{+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$$
$$\times\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0\mathrm{e}^{-\mathrm{i}k^0x^0}}{(\mathrm{i}w+\varepsilon-\mathrm{i}k^0)^{1-\alpha}(-\mathrm{i}w+\varepsilon-\mathrm{i}k^0)^{1-\alpha}} \tag{A2.1}$$

Using the result of [13, pp 320, 1059], we get

$$D_{\mathrm{ret}}^{\alpha}(x)=\int\frac{\mathrm{d}^2k}{(2\pi)^2}\mathrm{e}^{+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}2^{1/2-\alpha}\frac{\Gamma(\frac{3}{2}-\alpha)}{\Gamma(2-2\alpha)}$$
$$\times J_{1/2+\alpha}(\mathrm{e}^{\mathrm{i}\pi}wx^0)\theta(x^0)\mathrm{e}^{-\varepsilon x^0}\left(\frac{x^0}{w}\right)^{1/2-\alpha}. \tag{A2.2}$$

The angular integration is performed as shown above and the integral of the double Bessel functions is found in [14, p 210]. The result (equation (3.8)) then follows.

The advanced functions are obtained from the above results through the transformation $k\to -k$ and $x\to -x$.

Appendix 3

The $D_{+}^{\alpha}(x)$ function (equation (4.3)) may be calculated along the lines of the previous appendices or using the inverse Fourier transform to the $D_{\mathrm{ret}}^{\alpha}(x)$ functions and making $x\leftrightarrow k$ and $\alpha\to(\frac{1}{2})-\alpha$.

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